Intro to Crys. Coh.

Setup:
$$\begin{tikzcd} X_0 \arrow[r] \arrow[d] \arrow[dr, phantom, "\#"] & X \arrow[d] \\ S_0 \arrow[r] & S \end{tikzcd}$$
$S_0 = V(I)$
(S, I, γ) PD structure.

e.g. $S = \mathrm{Spec}\, W_n(k)$. k perfect.
$S_0 = \mathrm{Spec}\, k$.

- $(X_0/S)_{crys}$ site.
- $\mathcal{O}_{X_0/S} \in Sh((X_0/S)_{crys})$.
- P.L. $\mathcal{O}_{X_0/S} \xrightarrow[\text{qis.}]{\simeq} L(\Omega^\bullet_{X/S})$ X/S smooth as $\mathcal{O}_{X_0/S}$-mod
- $H^*_{crys}(X_0, \mathcal{O}_{X_0/S}) \cong H^*_{dR}(X/S)$.

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§§1. GM connection.

$$\begin{tikzcd} X \arrow[d] \\ S \end{tikzcd}$$
proper smooth $/\mathbb{C}$.

Thm $\mathcal{H}^*_{dR}(X/S) := R^*\pi_*(\Omega^\bullet_{X/S})$. has integrable connection

Infinitesimal interpretation

$$\begin{tikzcd} S(1) \arrow[r, hook] & S \times S \\ S \arrow[u] \arrow[ur, "\Delta"'] \end{tikzcd}$$
1-st order nbhd of Δ.

$S(1) \underset{p_2}{\overset{p_1}{\rightrightarrows}} S$. $p_1^* H^*_{dR}(X/S) \xrightarrow{\varphi} p_2^* H^*_{dR}(X/S)$.
isom. as $\mathcal{O}_{S(1)}$-modules

$$\begin{tikzcd} S'(1) \arrow[r] & S \times S \times S \\ S \arrow[u] \arrow[ur] \end{tikzcd}$$

$S'(1) \rightrightarrows S \times S \times S$.

$S'(1) \xrightarrow{p_{12}} S(1)$, $\xrightarrow{p_{13}}$, $\xrightarrow{p_{23}}$

$$\begin{tikzcd}
p_1^* H^*_{dR} \arrow[rr, "p_{13}^*(\varphi)"] \arrow[dr, "p_{12}^*(\varphi)"'] & & p_3^* H^*_{dR} \\
& p_2^* H^*_{dR} \arrow[ur, "p_{23}^*(\varphi)"'] &
\end{tikzcd}$$

$$\mathbb{C}[[t]]/(t^2) \longleftrightarrow W_2(k)$$

$$H^*_{dR}(\mathcal{X}/W_2) \cong H^*_{dR}(\mathcal{X}'/W_2) \qquad \mathcal{X}, \mathcal{X}' \text{ lifting of } X_0.$$

$$\text{\rotatebox{90}{$\cong$}}$$

$$H^*_{crys}(X_0/W_2).$$

Linear operator. $X \xrightarrow{\pi} S$ smooth.

$h : \mathcal{F} \to \mathcal{G}$ $\pi^{-1}\mathcal{O}_S$-linear.

linearization. $\bar{h}$: $\mathcal{O}_X \otimes_{\mathcal{O}_S} \mathcal{F} \longrightarrow \mathcal{G}$ assuming $\mathcal{G}$ has $\mathcal{O}_X$-mod. $\mathcal{F}$,

$$\begin{tikzcd}
\mathcal{O}_X \otimes_{\mathcal{O}_S} F \arrow[rr] & & \mathcal{G} \\
& F \arrow[ul] \arrow[ur, "h"'] &
\end{tikzcd}$$

e.g. $\Omega^i_{X/S} \xrightarrow{d} \Omega^{i+1}_{X/S}$. $\mathcal{P}$.

$$\mathcal{O}_X \otimes_{\mathcal{O}_S} F \cong \underbrace{(\mathcal{O}_X \otimes_{\mathcal{O}_S} \mathcal{O}_X)}_{=\mathcal{P}} \otimes_{\mathcal{O}_X} \mathcal{F},$$

Defn. h is of order $\le n$ iff.

$$\bar{h} : (\mathcal{O}_X \otimes_{\mathcal{O}_S} \mathcal{O}_X) \otimes_{\mathcal{O}_X} \mathcal{F} \longrightarrow \underbrace{(\mathcal{P}/I_\Delta^{n+1})}_{=\mathcal{P}_n} \otimes_{\mathcal{O}_X} \mathcal{F} \longrightarrow \mathcal{G}.$$

$\mathcal{O}_X$-linear.

such $\bar{h}$, if exist, is unique.

e.g. $n=0$, h IS $\mathcal{O}_X$-linear.

$n=1$ $\mathcal{P}_1 = \mathcal{O}_X \oplus \Omega^1_{X/S}$.

e.g.
$$\begin{array}{ccccc}
\mathcal{O}_X & \xrightarrow{d} & \mathcal{P}_1 & = & \mathcal{O}_X \oplus \Omega^1_{X/S} \\
x & \longmapsto & 1\otimes x & & \\
x & \longmapsto & & & (x\otimes 1,\ 1\otimes x - x\otimes 1).
\end{array}$$

Prop. X/S smooth w/ coordinates $x_1, \cdots, x_n$

Let $\xi_i = 1\otimes x_i - x_i \otimes 1$.

$\Rightarrow$ $\mathcal{P}_{X/S,m}$ is free $\mathcal{O}_X$-mod.

basis $\{ \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n} \mid \sum \alpha_i \le m \}$.

as left or right $\mathcal{O}_X$-mod.

composition:
$$\begin{tikzcd}
\mathcal{F} \arrow[r, "f"] \arrow[d] \arrow[dr] & \mathcal{G} \arrow[r, "g"] \arrow[dr] & \mathcal{H} \\
 & \mathcal{P}^n \otimes_{\mathcal{O}_X} \mathcal{F} \arrow[u] \arrow[d] & \mathcal{P}_m \otimes_{\mathcal{O}_X} \mathcal{G} \arrow[u] \\
\mathcal{P}_{n+m} \otimes_{\mathcal{O}_X} \mathcal{F} \arrow[r] & \mathcal{P}_m \otimes_{\mathcal{O}_X} \mathcal{P}_n \otimes_{\mathcal{O}_X} \mathcal{F} \arrow[ur, "\mathrm{Id}_{\mathcal{P}_m} \otimes_{\mathcal{O}_X} \bar{f}"'] &
\end{tikzcd}$$

where
$$\begin{tikzcd}
a\otimes b \arrow[r, mapsto] & a\otimes 1 \otimes b \\
\mathcal{O}_X \otimes_{\mathcal{O}_S} \mathcal{O}_X \arrow[r] \arrow[d, equal] & \mathcal{O}_X \otimes_{\mathcal{O}_S} \mathcal{O}_X \otimes_{\mathcal{O}_S} \mathcal{O}_X \arrow[d, equal] \\
\mathcal{P} \arrow[r, "\delta"] \arrow[d] & \mathcal{P} \otimes_{\mathcal{O}_X} \mathcal{P} \arrow[d] \\
\mathcal{P}_{n+m} \arrow[r, "\delta_{m,n}"] & \mathcal{P}_m \otimes \mathcal{P}_n
\end{tikzcd}$$

b/c $\boxed{\begin{array}{l} \delta(\xi) = \mathbb{1}\otimes \xi + \xi \otimes \mathbb{1}. \\ \text{if } \xi = 1\otimes x - x \otimes 1,\ \mathbb{1} = 1\otimes 1 \end{array}}$

Defn. $\mathcal{D}iff_{X/S}(\mathcal{F}, \mathcal{G}) = \varinjlim_n \mathcal{D}iff^{\le n}(\mathcal{F}, \mathcal{G})$. sheaf of germs of differential operators of order $\le n$ between $\mathcal{F}$ & $\mathcal{G}$.

Thm ① $\mathcal{D}iff^{\le n}_{X/S}(\mathcal{F}, \mathcal{G}) = \mathcal{H}om_X(\mathcal{P}^n \otimes_{\mathcal{O}_X} \mathcal{F}, \mathcal{G})$

$\mathcal{O}_X$-structure $\alpha \in \mathcal{O}_X$ $\alpha\cdot h := \mathcal{F} \xrightarrow{h} \mathcal{G} \xrightarrow{\cdot\alpha} \mathcal{G}$

$\mathrm{Diff}_{X/S}(\mathcal{F}, -)$ is pro-represented by $\{\mathcal{P}_n \otimes \mathcal{F}\}$.

② $\mathrm{Diff}^{\leq n}_{X/S}(\mathcal{O}_X, \mathcal{O}_X)$ locally free.

③ $\mathrm{Diff}_{X/S}(\mathcal{O}_X, \mathcal{O}_X)$ is a sheaf of rings $(\subseteq \mathcal{E}nd(\mathcal{O}_X))$.

$a \in \mathcal{O}_X$, $\underset{a\circ h - h\circ a}{\underset{\shortparallel}{\mathrm{ad}_a h}} \in$ order $\leq n-1 \iff h \in$ order $\leq n$.

Almost linearization.

$F \xrightarrow{h} G$. $\pi^{-1}\mathcal{O}_S$-linear.

$$\begin{tikzcd} \mathcal{O}_X \otimes_{\mathcal{O}_S} F \arrow[r] \arrow[d, equal] & \mathcal{O}_X \otimes_{\mathcal{O}_S} G \arrow[d, equal] \\ \mathcal{P} \otimes_{\mathcal{O}_X} F \arrow[d, "\delta \otimes \mathrm{id}_F"] & \mathcal{P} \otimes_{\mathcal{O}_X} G \\ \mathcal{P} \otimes_{\mathcal{O}_X} \mathcal{P} \otimes_{\mathcal{O}_X} F \arrow[ur, "\mathrm{id}_{\mathcal{P}} \otimes \bar{h}"'] & \end{tikzcd}$$

$L : \{\mathcal{O}_X\text{-mod. diff operators}\}$

$\downarrow$

$\{$stratified $\mathcal{O}_X$-mod w/ horizontal $\mathcal{O}_X$-linear maps$\}$

stratified $\longleftrightarrow$ descent datum.

1-connection: (E, ∇) $\nabla : E \longrightarrow E \otimes \Omega^1_{X/S}$.

$\iff$ $\mathcal{P}_1$-linear isom. $\varepsilon : \mathcal{P}_1 \otimes_{\mathcal{O}_X} E \longrightarrow E \otimes_{\mathcal{O}_X} \mathcal{P}_1$

s.t. modulo $\ker(\mathcal{P}_1 \to \mathcal{O}_X)$, reduces to Id_E.

$\theta : E \longrightarrow \mathcal{P}_1 \otimes E \xrightarrow{\varepsilon} E \otimes \mathcal{P}_1$.

$\nabla(s) = \theta(s) - s \otimes \mathbb{1}$.

e.g. $E = \mathcal{O}_X$. $\nabla = d$ $\mathcal{O}_X \longrightarrow \Omega^1_X$. $\mathcal{P}^1 \xrightarrow{\varepsilon = \mathrm{Id}} \mathcal{P}^1$

$\{n\text{-connections}\} = \{\text{stratifications}\}$.

$\forall\ n \in \mathbb{N}_{\geq 0}$ $\quad \varepsilon_n : \mathcal{P}_{X/S,n} \otimes_{\mathcal{O}_X} E \xrightarrow{\sim} E \otimes_{\mathcal{O}_X} \mathcal{P}_{X/S,n}$

1) ε_n - $\mathcal{P}_{X/S,n}$ -linear.

2)
$$\begin{array}{ccc} \mathcal{P}_{X/S,n} \otimes_{\mathcal{O}_X} E & \xrightarrow{\varepsilon_n} & E \otimes_{\mathcal{O}_X} \mathcal{P}_{X/S,n} \\ \downarrow & & \downarrow \\ \mathcal{P}_{X/S,m} \otimes_{\mathcal{O}_X} E & \xrightarrow{\varepsilon_m} & E \otimes_{\mathcal{O}_X} \mathcal{P}_{X/S,m} \end{array} \qquad \forall\ n \geq m.$$

3). $\varepsilon_0 = Id_E$.

4) $\mathcal{P}_{X/S,n}(2)$ is the n-th inf. nbhd of $X \hookrightarrow X \times_S X \times_S X$.

$$\mathcal{P}_{X/S,n}(2) \xrightarrow{p_{ij}} \cancel{X \times_S X}\ \mathcal{P}_{X/S,n}$$

$$p_{23}^*(\varepsilon_n) \circ p_{12}^*(\varepsilon_n) = p_{13}^*(\varepsilon_n).$$

Prop. X/S smooth, E is $\mathcal{O}_X$-mod. TFAE:

1) a stratification on E

2) a collection of $\Theta_n : E \longrightarrow E \otimes_{\mathcal{O}_X} \mathcal{P}_{X/S,n}$.

$$\begin{array}{ccc} E & \longrightarrow & E \otimes_{\mathcal{O}_X} \mathcal{P}_{X/S,n} \\ & \searrow \quad \nearrow & \\ & \mathcal{P}_{X/S,n} \otimes_{\mathcal{O}_X} E & \end{array}$$

3). An $\mathcal{O}_X$-linear ring hom.

$$\mathrm{Diff}_{X/S}(\mathcal{O}_X, \mathcal{O}_X) \longrightarrow \mathrm{Diff}_{X/S}(E, E).$$

3') $\nabla : \mathrm{Diff}_{X/S}(\mathcal{F}, \mathcal{G}) \longrightarrow \mathrm{Diff}_{X/S}(E \otimes_{\mathcal{O}_X} \mathcal{F}, E \otimes_{\mathcal{O}_X} \mathcal{G})$

$$\begin{array}{ccc} E \otimes_{\mathcal{O}_X} \mathcal{F} & \longrightarrow & E \otimes_{\mathcal{O}_X} \mathcal{G} \\ & {\scriptstyle \Theta_n \otimes Id_{\mathcal{F}}} \searrow \quad \nearrow & \\ & E \otimes_{\mathcal{O}_X} \mathcal{P}_{X/S,n} \otimes_{\mathcal{O}_X} \mathcal{F} & \end{array}$$

4), A "crystal" is a cptible family of sheaves

E_T, $\mathcal{O}_T$-module for $U \hookrightarrow T$ nilpotent ideal.

$U \subset X$ open

w/ transitive isom. $u^* E_T \longrightarrow E_{T'}$.

$$(U', T') \longrightarrow (U, T).$$

$$\begin{tikzcd} T' \arrow[r] & T \\ U' \arrow[u] \arrow[r] & U \arrow[u] \end{tikzcd}$$

pf. (1) $\Rightarrow$ (4). assuming U, T are affine.

$$\begin{tikzcd} U \arrow[r, hook] & T \arrow[l, bend right, dashed, "g"'] \arrow[l, bend left, dashed, "h"] \\ X \arrow[u, phantom, "\cap"] \end{tikzcd}$$

$E_T := g^* E$

independence of section: $T \xrightarrow{(g,h)} X \times_S X$. ε_m gives $g^*E \simeq h^*E$.

$$\begin{tikzcd} T \arrow[r, "(g{,}h)"] \arrow[dr] & X \times_S X \\ & P_{X/S,m} \arrow[u] \end{tikzcd}$$

hence a well-defined crystal.

Defn. Infinitesimal site: X/S, obj $= (U, T)$. morphism = comm. diag.

covering = jointly covering $|T|$.

$\mathcal{O}_{X/S}((U,T)) = \mathcal{O}_T$. a natural crystal.

$(\Omega^{\cdot}_{X/S})$ is a cplx of coherent sheaves.

slogan: use linearization to turn $(\Omega^{\cdot}_{X/S})$ into cplx of crystals.

$$\mathcal{O}_{X/S} \xrightarrow{q.is} L(\Omega^{\cdot}_{X/S}).$$

aL has almost stratification:

$$aL(E) = \mathcal{O}_X \underset{\mathcal{O}_S}{\otimes} E \longrightarrow E \underset{\mathcal{O}_S}{\otimes} \mathcal{O}_X \longrightarrow \mathcal{O}_X \underset{S}{\otimes} E \underset{S}{\otimes} \mathcal{O}_X$$

$$(a \otimes x) \longmapsto (x \otimes a).$$

$$\begin{array}{ccc} \mathcal{P}_{X/S} \underset{X}{\otimes} E & \xrightarrow[\mathcal{O}_X\text{-linear}]{\theta_E} & \mathcal{P}_{X/S} \underset{X}{\otimes} E \underset{X}{\otimes} \mathcal{P}_{X/S}. \\ \| & & \| \\ aL(E) & & aL(E) \underset{X}{\otimes} \mathcal{P}_{X/S}. \end{array}$$

$$\Rightarrow \varepsilon : \mathcal{P} \underset{X}{\otimes} aL(E) \xrightarrow{\simeq} aL(E) \underset{X}{\otimes} \mathcal{P}. \quad \mathcal{P}\text{-linear}.$$

$\mathcal{F} \xrightarrow{h} \mathcal{G}$ $\quad aL(h) = \mathcal{O}_X \underset{S}{\otimes} h.$

$$\nabla(h) : \mathcal{P} \underset{X}{\otimes} E \xrightarrow{\delta \otimes id_E} \mathcal{P} \underset{X}{\otimes} \mathcal{P} \underset{X}{\otimes} E.$$

$$\mathcal{P} \underset{X}{\otimes} \mathcal{P} \underset{X}{\otimes} E \xrightarrow{id_{\mathcal{P}} \otimes \bar{h}} \mathcal{P} \underset{X}{\otimes} \mathcal{F}$$

$$aL(h) = \nabla(h).$$

$$\begin{array}{ccc} aL(E) & \xrightarrow{\theta_E} & aL(E) \otimes \mathcal{P} \\ \nabla(h) \downarrow & \circlearrowright & \downarrow \nabla(h) \otimes id_{\mathcal{P}} \\ aL(F) & \xrightarrow{\theta_F} & aL(F) \otimes \mathcal{P} \end{array}$$

Defn. $L.(E) := \varprojlim_n \{ \mathcal{P}^n \underset{\mathcal{O}_X}{\otimes} E \}$. Now if h is a diff. operator

$L.(h) := \nabla(h)$.

$$\mathcal{P}_{m+n} \underset{X}{\otimes} E \xrightarrow{\delta_{m,n} \otimes id_E} \mathcal{P}_m \underset{X}{\otimes} \mathcal{P}_n \underset{X}{\otimes} E \xrightarrow{id_{\mathcal{P}_m} \underset{X}{\otimes} \tilde{h}} \mathcal{P}_m \underset{X}{\otimes} \mathcal{F}.$$

§§2. PD structure.

Defn. (A, I, γ) $\gamma_i : I \longrightarrow A$ s.t. $\gamma_i(x) \in I$ if $i \geq 1$...

$(A, I, \gamma) \longrightarrow (B, J, \delta)$.

e.g. DVR. w/ $p = u \cdot \pi^e$ w/ $e \leq p-1$ $\forall$ prime p.

Lemmata. • if $m \cdot A = 0$ for some $m \in \mathbb{N}$ $\Rightarrow$ I is nilpotent ideal. (A, I, γ).

• if $J \subseteq I$ PD subideal. ~~then~~ $(\gamma_r(j) \in J, \forall j \in J)$

Then $(A/J, I/J, \bar{\gamma})$ is also a PD structure.

& $A \twoheadrightarrow A/J$ is a PD morphism.

• ~~colimit~~ direct limit of PD w/ PD morphism. is still PD.

• If $\begin{array}{c} A \longrightarrow B \\ \downarrow \\ C \end{array}$, $I \subseteq B$, $J \subseteq C$, s.t. $B \rightleftarrows B/I$ $C \rightleftarrows C/J$

then $K = \ker(B \otimes_A C \twoheadrightarrow B/I \otimes_A C/J)$

has a unique PD structure cptible w/ (B, I) & (C, J).

• I is a PD ideal $\Rightarrow$ $I^n \subseteq I$ PD subideal.

e.g. $(W(k), (p)) \Rightarrow (W_n(k), \overline{(p)})$ has a ~~unique~~ natural PD.

• A. ring. M. A-mod.

~~Prop.~~ $\Gamma_A(M) = A \oplus M \oplus \Gamma_2(M) \oplus \cdots$

has P.D. $(\Gamma_A(M), \Gamma_A^+(M), \gamma)$.

$M = \bigoplus_I A \cdot x_i$. $\Gamma_A(M)$ is divided poly. in $\gamma_n(x_i)$.

Defn. ① (A,I,γ). $A \to B$. γ <u>extends</u> to B if $\exists$ $\bar{\gamma}$ on $I\cdot B$ s.t.

$$(A,I,\gamma) \to (B, IB, \bar{\gamma}) \text{ is PD.}$$

② (A,I,γ), (B,J,δ) $A \to B$ <u>cptible w/ PD structure</u> if
γ extends to $\bar{\gamma}$ & $\bar{\gamma} = \delta$ on $IB \cap J$.

i.e. $\exists$ α PD structure on $IB + J$ s.t.

$$\begin{tikzcd}
(A,I,\gamma) \arrow[r] & (B, IB+J, \alpha) \\
& (B,J,\delta) \arrow[u]
\end{tikzcd}$$

Thm. (PD envelope) (A,I,γ) PD. $A \to B$. $J \subseteq B$, ideal, Then $\exists!$ a B-alg. $D_{B,\gamma}(J)$ w/ PD ideal $(\bar{J}, [n])$.
s.t. $J \cdot D_{B,\gamma}(J) \subseteq \bar{J}$ and $[\cdot]$ is cptible w/

$$(A,I,\gamma) \longrightarrow (D_{B,\gamma}(J), \bar{J}, [\cdot]).$$

universal property

$$\begin{tikzcd}
& (D_{B,\gamma}(J), \bar{J}, [\cdot]) \arrow[dr, dashed, "\text{PD morphism}"] & \\
(B,J) \arrow[ur] \arrow[rr, "J \mapsto K"] & & (C, K, \delta) \\
& (A,I,\gamma) \arrow[ul] \arrow[ur, "\text{cptible}"'] &
\end{tikzcd}$$

Rmk.
- $D_{B,\gamma}(J)$ is generated by $\{x^{[n]}, n \geq 0, x \in J\}$ as B-alg.
  $D_{B,\gamma}(J) \cong \Gamma_B(J)/\sim$.
- $\bar{J}$ is generated by image $\Gamma_B^{+}(J)$.
- if γ extends to B/J, then $D_{B,\gamma}(J)/\bar{J} \cong B/J$.

- If $B=A[x_1,\cdots,x_n]$, $J=(x_1,\cdots,x_n)$.
  Then $D_{B,0}(J)=A\{x_1,\cdots,x_n\}$.
- If γ extends to B/J, and $B\twoheadrightarrow B/J$ has a splitting as A-alg. (e.g. B/J is smooth $/A$).
  Then $D_{B,\gamma}(J)\cong D_{B,0}(J)$ ~~w/ str~~ as PD structures.

Prop. Let S be a scheme, $I\subseteq\mathcal{O}_S$ quasi-coh. w/ PD structure γ. and let X be an S-scheme.
~~Then~~ If $\mathcal{B}$ is a qcoh $\mathcal{O}_X$-alg. & $J\subseteq\mathcal{B}$ qcoh ideal. then $\exists!\ D_{\mathcal{B},\gamma}(J)$ a qcoh $\mathcal{O}_X$-alg.

(S, I, γ)

$$\begin{tikzcd} X \arrow[r, hook] \arrow[dr] & Y \arrow[d] \\ & S \end{tikzcd}$$ closed immersion

Defn. $J=V(X)$. $\qquad D_{X,\gamma}(Y):=\underline{\mathrm{Spec}}_Y\, D_{\mathcal{O}_Y,\gamma}(J)$.

So if γ extends to X, then

$$\begin{tikzcd} & X \arrow[r, hook, "j"] \arrow[dr] & D_{X,\gamma}(Y) \arrow[d] \\ X' \arrow[ur] \arrow[r, hook] & Y' \arrow[r] \arrow[ur, dashed, "\exists!"] & Y \end{tikzcd}$$

and if closed w/ PD structure cptible w/ γ

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d] \arrow[dr, phantom, "\#"] & X \arrow[d] \\ S_0 \arrow[r, hook] & S \end{tikzcd}$$

now, assuming $S_0\hookrightarrow S$ PD thickening γ extends to X.

$$\begin{tikzcd} & X_0 \arrow[r] \arrow[dr] & D_{X_0,\gamma}(X) \arrow[d] \\ U \arrow[ur] \arrow[r, hook, "\text{PD thickening}"'] & T \arrow[r, dashed] \arrow[ur, dashed] \arrow[d] & X \\ & S & \end{tikzcd}$$

$\leftarrow$ if X/S smooth.

Prop. $\begin{tikzcd} & S & \\ X \arrow[ur] \arrow[rr, hook] & & Y \arrow[ul] \end{tikzcd}$ closed imm. of smooth S-schemes. $m \cdot \mathcal{O}_Y = 0$ for some $m \in \mathbb{N}$.

Then $D_{X,\gamma}(Y)$ is locally $\cong$ P.D. polynomial

$\parallel$ locally

$\mathcal{O}_X\{t_1, \dots, t_d\}$.

application: if $\begin{tikzcd} & D(1) \arrow[dr] & \\ X \arrow[ur] \arrow[rr, hook] \arrow[dr, "\text{sm.}"'] & & X \times_S X \arrow[dl] \\ & S & \end{tikzcd}$ $\quad D(1) \underset{\text{loc.}}{\cong} \mathcal{O}_X\{t_1, \dots, t_n\}$ $\quad \mathcal{P} \to D(1)$.

upshot: almost ~~⊗~~ linearization, should be $\otimes_{\mathcal{O}_X} D(1)$.

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assume $p^n \mathcal{O}_S = 0$ (☆). X_0/S. (S, I, γ) PD-scheme.
γ extends to X_0.

Defn: $(X_0/S)_{crys}$: obj: $\begin{tikzcd} U \arrow[r, hook] & T \arrow[d] \\ X_0 \arrow[u, phantom, "\cap\text{ open}"] \arrow[r] & S \end{tikzcd}$ where $U = V(J)$, $J \subseteq \mathcal{O}_T$. (T, J, δ) PD. cptible w/ γ.

☆ $\Rightarrow$ J is nil ideal. ($\forall j \in J$, $j^{p^n} = 0$).

morphism: $\begin{tikzcd} T' \arrow[r] & T \\ U' \arrow[u, hook] \arrow[r, hook, "\text{open}"] & U \arrow[u, hook] \end{tikzcd}$ $\quad (T', J', \delta') \longrightarrow (T, J, \delta)$ PD morphism.

fiber product???

covering: $\{T_i \longrightarrow T\}$ is an open imm. & jointly cover $|T|_{Zar}$.

Prop. ~~Defn.~~ A sheaf $\mathcal{F} \in Sh((X_0/S)_{crys}) \Leftrightarrow \forall (U, T, \delta) \longmapsto \mathcal{F}_T \in Zar(T)$ s.t.
$\forall u: (U', T', \delta') \longrightarrow (U, T, \delta)$ $\exists\, \rho_u : u^{-1}\mathcal{F}_T \longrightarrow \mathcal{F}_{T'}$ satisfying composition law.
and if $T' \xhookrightarrow{\text{open}} T$, then ρ_u is an isom.

e.g. 1) ~~$\mathcal{O}_T$~~. $(U, T, \delta) \longmapsto \mathcal{O}_T$, denoted $\mathcal{O}_{X_0/S}$

2) $(X_0)_{Zar} \xrightarrow{i_{X_0/S}} (X_0/S)_{crys}$ $\qquad (i_{X_0/S})_* \mathcal{O}_{X_0} : (U,T,\delta) \mapsto$
$U \longleftarrow (U,T,\delta)$

3) $0 \to J_{X/S} \to \mathcal{O}_{X_0/S} \to i_* \mathcal{O}_{X_0} \to 0$. exact.

$$J_{X/S}(U,T,S) = \ker(\mathcal{O}_T \twoheadrightarrow \mathcal{O}_U).$$

Rmk. • $Sh((X_0/S)_{crys})$ has enough points.

Prop. ① $(X_0/S)_{crys}$ has finite product / inverse limit. over finite ordered. index set.
(a.k.a., finite limits exist ?).

②
$$\begin{tikzcd} X_0' \arrow[r, "g"] \arrow[d] & X_0 \arrow[d] \\ S' \arrow[r] & S \end{tikzcd} \Longrightarrow g_{crys,*} : Sh(X_0'/S') \longrightarrow Sh(X_0/S).$$

PD morphism. morphism of topoi.

$$g_{crys}^{-1}((U,T,\delta))(U',T',\delta') = \text{PD-Hom}_g((U',T',\delta'),(U,T,\delta))$$
$$= \{h, \text{PD-morphisms}\}.$$

$$\begin{tikzcd} T' \arrow[r, dashed, "h"] & T \\ U' \arrow[u, hook] \arrow[r] & U \arrow[u, hook] \\ X_0' \arrow[u, phantom, "\cap"] \arrow[r, "g"] & X_0 \arrow[u, phantom, "\cap"] \end{tikzcd}$$

$$g_{crys*}(\mathcal{F})(U,T,\delta) = \text{Hom}(g_{crys}^{-1}(U,T,\delta), \mathcal{F}).$$

③ $(X_0'/S', \mathcal{O}_{X_0'/S'}) \longrightarrow (X_0/S, \mathcal{O}_{X_0/S})$.

namely $\mathcal{O}_{X_0/S} \longrightarrow g_{crys*} \mathcal{O}_{X_0'/S'}$.

given $f \in \mathcal{O}(T)$. want : $\tilde{f} \in \text{Hom}(g_{crys}^{-1}(U,T,\delta), \mathcal{O}_{X_0'/S'})$.

$$\tilde{f}|_{T'} \in \text{Hom}(\text{PD-Hom}_g(T',T), \mathcal{O}_{T'}).$$
$$\tilde{f}|_{T'}(\varphi) = \varphi^\#(f).$$

Thm $H^i((X/S)_{crys}, \mathcal{O}_{X/S}) \cong H^i((X_0/S)_{crys}, \mathcal{O}_{X_0/S})$.

$\boxed{\text{no smooth assumption!}}$

$$\begin{array}{ccc} X_0 & \hookrightarrow & X \\ \downarrow & & \downarrow \\ S_0 & \underset{V(I)}{\hookrightarrow} & S \end{array}$$

pf: $\begin{array}{ccc} X_0 & \xhookrightarrow{i} & X \\ \downarrow & & \downarrow \\ S & = & S \end{array}$

① $i_{crys}^{-1} T$ representable by $(U \cap X_0 = U_0, T, \bar{\delta})$ in $(X_0/S)_{crys}$.
$T = (U, T, \delta)$

② $i_{crys,*}$ is exact.

③ $i_{crys,*}(\mathcal{O}_{X_0/S}) \cong \mathcal{O}_{X/S}$.

④ Leray s.s. for $(\Gamma \circ i_{crys})$.

④: $\Gamma(F) = \mathrm{Hom}(e, F)$. $(\Gamma \circ i_*)(F) = \mathrm{Hom}(e, i_* F)$
$= \mathrm{Hom}(i^{-1} e, F) = \mathrm{Hom}(e, F) = \Gamma(F)$.

① $\Rightarrow$ ②, ③: $i_{crys,*}(\mathcal{O}_{X_0/S})(T) = \mathrm{Hom}(i^{-1} h_T, \mathcal{O}_{X_0/S})$
$= \mathcal{O}_{X_0/S}(T) = \mathcal{O}_T(T) = \mathcal{O}_{X/S}(T)$.

exactness follows similarly.

①: by defn. $(T, I + I_U, \bar{\delta})$.

Prop. • $\exists$ morphism of topoi: $u_{X_0/S} : (X_0/S)_{crys} \longrightarrow (X_0)_{Zar}$.

① $u_{X_0/S}(F)(U) = \Gamma((U/S)_{crys}, j_{crys}^* F)$ $\quad j : U \xhookrightarrow{\text{Zar. open}} X_0$.

② $u^{-1}_{X_0/S}(E)(U, T, \delta) = E(U)$.

smoothness $\rightarrow$ • $\mathcal{O}_{X_0/S} \xrightarrow[qis]{} L.(\Omega^{\cdot}_{X/S}) := D(1) \otimes_{\mathcal{O}_X} \Omega^{\cdot}_{X/S}$.
(Poincaré Lemma).

$$\begin{array}{ccc} X_0 & \hookrightarrow & X \\ \downarrow & & \downarrow \scriptstyle{sm.} \\ S_0 & \hookrightarrow & S \end{array}$$

$$\begin{tikzcd} & D(1) \arrow[d] \\ X \arrow[ur] \arrow[r, hook] & X \times_S X \end{tikzcd}$$

$D(1) \leftarrow$ PD hull. (has a stratification!)

the connecting morphisms are subtle.

when $\begin{array}{ccc} X_0 & \longrightarrow & X \\ \downarrow & & \downarrow \\ k & \hookrightarrow & W_n(k) \end{array}$, we have $D(1) \cong \mathcal{O}_X$?

- $Ru_{X/S,*}(L.(\Omega^{\cdot}_{X/S})) \cong \Omega^{\cdot}_{X/S}$.

explanation: given F, an $\mathcal{O}_X$-mod. q.coh.

why is $\underline{\underline{D(1) \otimes_X F}}$ a sheaf on $(X_0/S)_{crys}$?

b/c has HPD stratification $\Leftrightarrow$ is a crystal.

① a $\mathcal{O}_{X_0/S}$-mod sheaf on $(X_0/S)_{crys}$.

② $\rho_u^* F_T \xrightarrow{\simeq} F_{T'}$.

Defn. E has HPD if
- $\exists\ D(1) \otimes_X E \xrightarrow[\simeq]{\varepsilon} E \otimes_X D(1)$. $D(1)$-linear.
- $\varepsilon|_\Delta = Id$.
- $X \hookrightarrow X \times_S X \times_S X$ cocycle w.r.t. $D(2) \rightrightarrows D(1)$.

$$\begin{tikzcd} X \arrow[r, hook] \arrow[dr, bend right] & X \times_S X \times_S X \\ & D(2) \arrow[u] \end{tikzcd}$$

Lemma:

$$\begin{tikzcd} & U \arrow[r, hook] \arrow[dl, "\supseteq"'] & T \arrow[d, dashed, "g"] \\ X_0 \arrow[d] & & X \arrow[dll, "sm."] \\ S & & \end{tikzcd}$$

$E^{crys}(U, T, \delta) := g^* E(T)$.

conversely $E^{crys} \longmapsto E^{crys}|_X$.

Rmk. $D(1) \otimes_X \Omega^k_{X/S} \xrightarrow{\delta \otimes id} D(1) \otimes_X D(1) \otimes_X \Omega^k_{X/S}$

$\downarrow$

$D(1) \otimes_X \mathcal{P}^1 \otimes_X \Omega^k_{X/S} \longrightarrow D(1) \otimes_X \Omega^{k+1}$

$D(1)/\bar{J}^{[2]} \cong \mathcal{P}^1$.

HPD stratification and Crystal.

$(S, \mathcal{I}, \gamma)$

Defn A crystal on $(X_0/S)_{crys}$ is $\mathcal{O}_{X_0/S}$-mod $\mathcal{F}$ s.t. $\forall \begin{array}{c}(U', T', \delta)\\ \downarrow f\\ (U, T, \delta)\end{array}$ in $(X_0/S)_{crys}$, then $\rho_f^* \mathcal{F}_T \xrightarrow{\simeq} \mathcal{F}_{T'}$ is an isom.

Prop. If X/S smooth, • TFAE:

① crystal on $(X_0/S)_{crys}$.

② crystal on $(X/S)_{crys}$

③ $\mathcal{O}_X$-mod w/ HPD stratification on X/S. where $X_0 = X \times_S S_0$.

• $\mathrm{Hom}(-, (X_0, X, \mathcal{I}\mathcal{O}_X, \gamma)) =: \tilde{X}$. Then $\tilde{X} \to e \in Sh((X_0/S)_{crys})$ is a covering.

Hence we have a Čech-to-cohomology S.S. to compute $Ru_{X/S,*}$.

key Lemma (localization) of topos

$Z = (V, Z, \varepsilon) \in (X_0/S)_{crys}$. (unconditional).

$Sh(X_0/S)|_{\tilde{Z}} \xrightarrow{j_Z} Sh(X_0/S)$.

Assume $V = X_0$, then ① $j_{Z,*}$ is exact.

② $j_{Z,*}E$ is acyclic for $u_{X/S,*}$.

$E \in Sh(X_0/S)|_{\tilde{Z}}$.

Čech covering. $\underbrace{\tilde{X} \times \tilde{X} \times \tilde{X}}_{= D(2)} \equiv \underbrace{\tilde{X} \times \tilde{X}}_{= D(1)} \rightrightarrows \tilde{X} \to e$.

$\check{C}^{\bullet}_{\tilde{X}}(E) = (\cdot \to \cdots \to j_{\tilde{X}^{k+1},*} j^*_{\tilde{X}^{k+1}} E \to \cdots)$

by simplicial method we have $E \xrightarrow{qis} \check{C}^{\bullet}_{\tilde{X}}(E)$.

apply $Ru_{X/S,*}$ $\quad Ru_{X/S,*}E \xrightarrow{\sim} (\to (u_{X/S} \circ j_{\tilde{X}^{k+1}})_* j^*_{\tilde{X}^{k+1}} E \to \cdots)$.

Prop. $Ru_{X/S,*}E \xrightarrow{\sim} \check{C}A^{\bullet}_{\tilde{X}}(E)$. where $\check{C}A^{\bullet}_{\tilde{X}}(E) = E_{(X, D_X(X^{k+1}))}$.

$$\begin{tikzcd} D_X(X^{\nu+1}) \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & D_X(X^{\nu}) \arrow[l, shift left=1] \arrow[l, shift right=1] \\ & X \arrow[ul, leftrightarrow] \arrow[u] \end{tikzcd}$$

Poincaré Lemma.

Rmk $\quad Ru_{X/S,*}(\mathcal{O}_{X/S}) \xrightarrow{\sim} Ru_{X/S,*}(L.\Omega^{\bullet}_{X/S}) \xrightarrow[\text{key. Lemma}]{\sim} \check{C}A^{\bullet}_{\tilde{X}}(L.\Omega^{\bullet}_{X/S})$

$$\begin{tikzcd} \check{C}A^{\bullet}_{\tilde{X}}(L.\Omega^{\bullet}_{X/S}) \\ \Omega^{\bullet}_{X/S} \arrow[u, "\simeq \text{ simplicial method.}"'] \end{tikzcd}$$

Last step: $\quad \check{C}A^{\nu}_{\tilde{X}}(L.\Omega^{k}_{X/S}) = \big(\underbrace{L\Omega^{k}_{X/S}}_{D(1)\otimes_X \Omega^k_{X/S}}\big)(X, D(\nu))$. $\quad$ HPD-stratified.

$$\cong D(\nu)\otimes_X D(1)\otimes_X \Omega^k_{X/S} \cong D(\nu+1)\otimes_X \Omega^k_{X/S}.$$

Fixing k: $\quad \Omega^k_{X/S} \xrightarrow[\text{qis}]{\sim} \big(D(1)\otimes_X \Omega^k_{X/S} \to D(2)\otimes_X \Omega^k_{X/S} \to \cdots\big).$

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pf of key Lemma: $\quad (j_{Z,*}E)(U,T,\delta) = p_{T,*}(E_{p_Z})$. where

$$\begin{tikzcd} D_{\underline{u\Delta v},\delta,\varepsilon}(T\times_S Z) \arrow[r, "p_T"] \arrow[dr, "p_Z"'] & T \\ U \arrow[u] \arrow[ur, bend right] & Z \end{tikzcd}$$

1) p_T is homeomorphic

2) $E \to F \to G$ exact iff $E_T \to F_T \to G_T$ exact $\forall T$.

hence $1) + 2) \Rightarrow$ ①.

pf of ② by S.S.: $\quad E_2^{p,q} = R^p u_{X/S,*} R j^q_{Z,*}(E) \Rightarrow R^{p+q}(u_{X/S}\circ j_Z)(E)$.

$\quad\quad \| \text{ by ①}$

$\quad\quad R^p u_{X/S,*}\, j_{Z,*}(E)$.

$\Rightarrow$ It suffices to show $\quad R^p(u_{X/S}\circ j_{Z,*})(E) = 0 \quad \forall\ p > 0$.

$$\begin{tikzcd} u_{X/S}\circ j_Z : (X_0/S)_{crys}|_Z \arrow[r] \arrow[rrr, bend right, "\varphi"'] & (X_0/S)_{crys} \arrow[r] & (X_0)_{Zar} \simeq & (Z)_{Zar} \end{tikzcd}$$

$\varphi_*(F) = F|_Z \quad$ Hence φ_* is exact.

Rmk: this pf implies ~~proposition~~, computes $\quad (u_{X/S}\circ j_{X^k})_* j^*_{X^k}(E) = E(X, \tilde{X}^k)$.